Klein - Gordon eq^n in Electromagnetic field :-

Klein - Gordon gave an equation for a particle of charge 'e' in electromagnetic field.

In quantum mechanics, the properties of electromagnetic field are described by the potentials known as vector potential 'A' and scalar potential 'ϕ'.

In electromagnetic field the momentum and energy of the particle of charge 'e' are given by :-

$$\left.\begin{aligned} p &\rightarrow p - \frac{eA}{c} \\ E &\rightarrow E - e\phi \end{aligned}\right\} \quad (1)$$

From $E^2 = p^2c^2 + m^2c^4$

$$(E - e\phi)^2 = c^2\left(p - \frac{eA}{c}\right)^2 + m^2c^4$$

$$(E - e\phi)^2 = (cp - eA)^2 + m^2c^4 \quad (2)$$

Replacing operator E and p by $i\hbar\frac{\partial}{\partial t}$ and $-i\hbar\nabla$ respectively, we get Klein Gordon eq^n as

$$\boxed{\left(i\hbar\frac{\partial}{\partial t} - e\phi\right)^2\psi = \left[(-i\hbar c\nabla - eA)^2 + m^2c^4\right]\psi} \quad (3)$$

This is K-G eq^n in electromagnetic field.

Now,

$$\left(i\hbar\frac{\partial}{\partial t} - e\phi\right)^2\psi = \left[-\hbar^2\frac{\partial^2}{\partial t^2} - ie\hbar\frac{\partial\phi}{\partial t} - ie\hbar\phi\frac{\partial}{\partial t} + e^2\phi^2\right]\psi$$

$$= -\hbar^2\frac{\partial^2\psi}{\partial t^2} - ie\hbar\psi\frac{\partial\phi}{\partial t} - 2ie\hbar\phi\frac{\partial\psi}{\partial t} + e^2\phi^2\psi$$

$$(-i\hbar c\nabla - eA)^2\psi = i^2\hbar^2c^2\nabla^2\psi + iec\hbar\nabla\cdot(A\psi) + iec\hbar A\cdot\nabla\psi + e^2A^2\psi$$

$$= \left[-\hbar^2c^2\nabla^2 + ie\hbar c\nabla\cdot A + 2ie\hbar cA\cdot\nabla + e^2A^2\right]\psi$$

Substituting these values in eqn ③,

$$\boxed{\left[-\hbar^2\frac{\partial^2}{\partial t^2} - ie\hbar\frac{\partial\phi}{\partial t} - 2ie\hbar\phi\frac{\partial}{\partial t} + e^2\phi^2\right]\psi = \left[-\hbar^2c^2\nabla^2 + ie\hbar c\,\nabla\cdot A + 2ie\hbar c\,A\cdot\nabla + e^2A^2 + m^2c^4\right]\psi} \quad ④$$

To find connection b/w this above eqn & similar non-relativistic eqn Let us make the following substitution taking mc^2 as the rest energy.

$$\psi(r,t) = \psi'(r,t)\,e^{-imc^2t/\hbar}$$

$$\frac{\partial\psi}{\partial t} = \left(\frac{\partial\psi'}{\partial t} - \frac{imc^2}{\hbar}\psi'\right)e^{-imc^2t/\hbar}$$

$$\frac{\partial^2\psi}{\partial t^2} = \left[\frac{\partial^2\psi'}{\partial t^2} - \frac{2imc^2}{\hbar}\cdot\frac{\partial\psi'}{\partial t} - \frac{m^2c^4}{\hbar^2}\psi'\right]e^{-imc^2t/\hbar}$$

substituting these values in ④, we get

$$\left[-\hbar^2\frac{\partial^2\psi'}{\partial t^2} + 2imc^2\hbar\frac{\partial\psi'}{\partial t} + m^2c^4\psi' - ie\hbar\frac{\partial\phi}{\partial t}\psi' \cancel{- ie\hbar\frac{\partial\phi}{\partial t}\psi'} - 2ie\hbar\phi\frac{\partial\psi'}{\partial t} - 2mc^2e\phi\psi' + e^2\phi^2\psi'\right]e^{-imc^2t/\hbar} = \left[-\hbar^2c^2\nabla^2 + ie\hbar c\,\nabla\cdot A + 2ie\hbar c\,A\cdot\nabla + e^2A^2 + m^2c^4\right]\psi' e^{-imc^2t/\hbar}$$

cancelling out the common factor $e^{-imc^2t/\hbar}$, $m^2c^4\psi'$ on both sides and dividing throughout by $2mc^2$, we get

$$-\frac{\hbar^2}{2mc^2}\frac{\partial^2\psi'}{\partial t^2} + i\hbar\frac{\partial\psi'}{\partial t} - \frac{ie\hbar}{2mc^2}\frac{\partial\phi}{\partial t}\psi' - \frac{ie\hbar\phi}{mc^2}\frac{\partial\psi'}{\partial t} - e\phi\psi' + \frac{e^2\phi^2}{2mc^2}\psi'$$

$$= \left[-\frac{\hbar^2}{2m}\nabla^2 + \frac{ie\hbar}{2mc}\nabla\cdot A + \frac{ie\hbar}{mc}A\cdot\nabla + \frac{e^2A^2}{2mc^2}\right]\psi'$$

Keeping in mind that rest energy $mc^2 >>$ non relativistic energy E and $mc^2 >> e\phi$, we may neglect the terms of order $1/mc^2$ as compared to E' and $e\phi$ and rearranging

$$i\hbar\frac{\partial\psi'}{\partial t} = \left[-\frac{\hbar^2}{2m}\nabla^2 + \frac{ie\hbar}{2mc}\nabla\cdot A + \frac{ie\hbar}{mc}A\cdot\nabla + \frac{e^2A^2}{2mc^2} + e\phi\right]\psi'$$

This is simply non-relativistic Schroedinger eq^n for a particle of charge e in electromagnetic field. The Klein-Gordon eq^n in electromagnetic field reduces to correct non-relativistic limit with appropriate approximations.

"The eq^n (4) is K-G eq^n in electromagnetic field but
this eq^n is not relativistic eq^n in actual sense. The reason of this is that :-

1. This eq^n is not applicable to electrons since, it involves only a single wave function. This means that it does not include the electron spin since we know for the description of a particle of spin $1/2$ atleast two wave functions are needed.

2. Since, the electron has two types of motion — orbital as well as spin and therefore spin-orbit interaction exists and it is of relativistic nature. and hence, two wave functions are automatically needed.

3. This eq^n could not explain the fine structure of H-atom.

4. However, this eq^n explains of zero-spin particles like mesons.

The above discrepancy were due to the following facts :-

(i) This eqⁿ is of II-order in time and so it gives I eign

(ii) $\Psi\Psi^*$ is not the probability density of the states because it is time dependent.

(iii) It does not satisfy the condition of orthogonal and normalization.

Besides this, there are other difficulties even then this eqⁿ is important for the description of non-spin particle